A note on the recurrence of edge reinforced random walks

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Abstract. We give a short proof of Theorem 2.1 from [MR07], stating that the linearly edge reinforced random walk (ERRW) on a locally finite graph is recurrent if and only if it returns to its starting point almost surely. This result was proved in [MR07] by means of the much stronger property that the law of the ERRW is a mixture of Markov chains. Our proof only uses this latter property on *finite* graphs, in which case it is a consequence of De Finetti's theorem on exchangeability.

Although the question of the recurrence of linearly edge reinforced random walks (ERRW) on infinite graphs has known important breakthroughs in the recent years (cf. notably [MR09]), it seems that the only known proof that one almost-sure return implies the recurrence of the walk is based on the difficult fact that ERRWs on infinite graphs are mixtures of Markov chains (cf. [MR07]). We provide in this note a short and simple proof of that property, with the finite case as the only tool.

Let $G = (V, E)$ be a locally finite undirected graph, and $\alpha = (\alpha_e)_{e \in E}$ be a family of positive real numbers. The *linearly edge reinforced random walk on G with initial weights α starting at $o \in V$* is the nearest-neighbour random walk $(X_k)_{k \geq 0}$ on V defined as follows: $X_0 = o$; then, at each step, the walk crosses a neighbouring edge chosen with a probability proportional to its weight; and the weight of an edge is increased by 1 after it is traversed.

The only property to be used in this note is the following consequence of De Finetti's theorem for Markov chains (cf. [DF80], and [KR99] for instance): if G is finite, then there exists a probability measure μ on transition matrices on G such that the law of the ERRW X is $\int P_\omega(\cdot) d\mu(\omega)$ where P_ω is the law of the Markov chain on V with transition ω starting at o.

Here is the statement of the (main) part of Theorem 2.1 in [MR07] (cf. remark after the proof).

Theorem – For the linearly edge-reinforced random walk (ERRW) on any locally finite weighted graph, the following two statements are equivalent:

(i) the ERRW returns to its starting point with probability 1;

(ii) the ERRW returns to its starting point infinitely often with probability 1.

Proof. On finite graphs, this result follows from a Borel-Cantelli argument (cf. [KR99] and the remark after the proof). Let us therefore denote by $\mathbb{P}$ the law of the ERRW on an *infinite* locally finite weighted graph G starting at o. Assume that condition (i) holds.

For any $n \in \mathbb{N}$, we introduce the finite graph G_n defined from the ball $B(n+1)$ of center o and radius $n+1$ in G by identifying the points at distance $n+1$ from o to a new point δ_n. The law of the ERRW on G_n (with same weights as in G) starting at o is denoted by $\mathbb{P}_{G_n}$.

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Let us also define the successive return times $\tau^{(1)}, \tau^{(2)}, \dots$ of the ERRW at o, the exit time T_n from $B(n)$, and the hitting time τ_{δ_n} of δ_n in G_n. Note that the laws $\mathbb{P}$ and $\mathbb{P}_{G_n}$ may be naturally coupled in such a way that the trajectories coincide up to time $T_n = \tau_{\delta_n}$.
We have, for all $k \geq 1$,

$$\mathbb{P}(\tau^{(k)} < \infty) = \mathbb{P}(\tau^{(k)} < \infty, \tau^{(k)} < T_n) + \mathbb{P}(T_n < \tau^{(k)} < \infty),$$

and the second term converges to 0 when $n \to \infty$ since $T_n \geq n \underset{n}{\longrightarrow} \infty$. Therefore,

$$\begin{aligned} \mathbb{P}(\tau^{(k)} < \infty) &= \mathbb{P}(\tau^{(k)} < \infty, \tau^{(k)} < T_n) + o_n(1) \\ &= \mathbb{P}_{G_n}(\tau^{(k)} < \infty, \tau^{(k)} < \tau_{\delta_n}) + o_n(1). \end{aligned} \tag{1}$$

(NB: the condition $\tau^{(k)} < \infty$ on last line could be dropped since (ii) is true for ERRW on finite graphs). In particular, assumption (i) gives:

$$\lim_n \mathbb{P}_{G_n}(\tau^{(1)} < \infty, \tau^{(1)} < \tau_{\delta_n}) = \mathbb{P}(\tau^{(1)} < \infty) = 1. \tag{2}$$

Since G_n is finite, we may write $\mathbb{P}_{G_n}$ as a mixture of Markov chains: $\mathbb{P}_{G_n}(\cdot) = \int P_{G_n,\omega}(\cdot) d\mu_n(\omega)$. Thus we have, according to (2),

$$\lim_n \int P_{G_n,\omega}(\tau^{(1)} < \infty, \tau^{(1)} < \tau_{\delta_n}) d\mu_n = 1$$

and, for all $k \geq 1$, according to (1) and Markov property (applied $k-1$ times),

$$\begin{aligned} \mathbb{P}(\tau^{(k)} < \infty) &= \lim_n \int P_{G_n,\omega}(\tau^{(k)} < \infty, \tau^{(k)} < \tau_{\delta_n})\, d\mu_n \\ &= \lim_n \int P_{G_n,\omega}(\tau^{(1)} < \infty, \tau^{(1)} < \tau_{\delta_n})^k\, d\mu_n. \end{aligned}$$

We may conclude that the last limit equals 1 thanks to the following very simple Lemma:

LEMMA. – If $(f_n)_n, (\mu_n)_n$ are respectively a sequences of measurable functions and probability measures such that, for all n, $0 \leq f_n \leq 1$, and $\int f_n d\mu_n \underset{n}{\longrightarrow} 1$, then:

$$\text{for every integer } k \geq 1, \quad \int (f_n)^k d\mu_n \underset{n}{\longrightarrow} 1.$$

PROOF. Indeed, we have $0 \leq f_n^k \leq 1$, hence:

$$0 \leq 1 - \int f_n^k d\mu_n = \int (1 - f_n^k) d\mu_n = \int (1 - f_n)(1 + f_n + \cdots + f_n^{k-1}) d\mu_n \leq k \int (1 - f_n) d\mu_n \underset{n}{\longrightarrow} 0.$$

□

As a conclusion, $\mathbb{P}(\tau^{(k)} < \infty) = 1$ for all $k \geq 1$, hence $\mathbb{P}(\forall k, \tau^{(k)} < \infty) = 1$, which is (ii). □

Remark Condition (ii) implies that the ERRW visits every edge in the connected component of the starting point infinitely often in both directions, by means of the conditional Borel-Cantelli lemma, cf. the end of the proof of Theorem 1.1 in [MR09] or Proposition 1 of [KR99] for a direct proof.

References

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